1 a) Seja $a_n = \dfrac{-2}{(2n+5)(2n+3)}$ (série telescópica)

Decompondo em frações parciais:

$$\frac{-2}{(2n+5)(2n+3)} = \frac{A}{(2n+5)} + \frac{B}{(2n+3)} = \frac{2An + 3A + 2nB + 5B}{(2n+5)(2n+3)}$$

Assim, $\begin{cases} 3A + 5B = -2 \\ \cancel{2}A + \cancel{2}B = 0 \end{cases} \Rightarrow A = -B \Rightarrow 3(-B) + 5B = -2 \Rightarrow \boxed{B = -1}$

Então $\boxed{A = +1}$

Logo

$$\sum_{n=1}^{\infty} + a_n = \sum_{n=1}^{\infty} \frac{1}{2n+5} - \frac{1}{(2n+3)}$$

$$= \left(\cancel{\frac{1}{7}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{11}} - \cancel{\frac{1}{9}}\right) + \cdots + \left(\frac{1}{2n+5} - \cancel{\frac{1}{2n+3}}\right) + \cdots$$

$$S_n = \sum_{n=1}^{\infty} a_n = -\frac{1}{5} + \frac{1}{2n+5}$$

Dessa forma, $\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} \left(-\frac{1}{5} + \frac{1}{2n+5}\right) = -\frac{1}{5}$

$$\sum_{n=1}^{\infty} a_n = -\frac{1}{5}$$

1) b) $\sum_{n=1}^{\infty} n e^{-n^2}$, (Teste da Integral)

$f(x) = x \cdot e^{-x^2}$, pelo teste da integral, pois

1) $f(x)$ é contínua $\forall x \in \mathbb{R}$

2) $f(x) = \frac{x}{e^{-x^2}}$ é decrescente $\forall x \geq 1$ $f(x) < 0$

3) $f(x)$ é positiva $\forall x \geq 1$.

Então,

$$\int_0^{\infty} x \cdot e^{-x^2} dx = \lim_{b \to \infty} \int_0^b x e^{-x^2} dx =$$

Integral por substituição : $u = x^2 \Rightarrow du = 2x\, dx \Rightarrow \frac{du}{2} = x\, dx$

Logo

$$\int_0^{\infty} x e^{-x^2} dx = \lim_{b \to \infty} \int_0^{b^2} \frac{1}{2} e^{-u} du = \lim_{b \to \infty} \frac{1}{2} \left[-e^{-u} \right]_0^{b^2} =$$

$$= \lim_{b \to \infty} -\frac{1}{2} \left[e^{-b^2} - 1 \right] = \frac{1}{2}$$

Integral indefinida converge, logo

a série $\sum_{n=1}^{\infty} n e^{-n^2}$ converge.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n+3}$.

Teste da comparação. Note que $a_n = \frac{1}{4^n+3}$

Considerando $b_n = \frac{1}{4^n}$, temos que

$$\frac{1}{4^n+3} < \frac{1}{4^n} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4^n+3} < \sum \frac{1}{4^n}$$

Sendo $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$ uma série geométrica com razão $r = \frac{1}{4} < 1$, logo convergente.

Pelo teste da comparação a série $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n+3}$ converge.

---

(3) a) $a_n = tg\left(\frac{2n\pi}{1+8n}\right)$. A sequência a_n é convergente se $\lim_{n\to\infty} a_n = L$

Temos $\displaystyle\lim_{n\to\infty} tg\left(\frac{2n\pi}{1+8n}\right) = \lim_{n\to\infty} tg\left(\frac{2\pi}{\frac{1}{n}+8}\right) =$

$= \displaystyle\lim_{n\to\infty} tg\left(\frac{\pi}{4}\right) = 1.$ Logo a_n converge.

---

b) $a_n = \frac{\cos^2 n}{2^n}$.

$$\lim_{n\to\infty} \frac{\cos^2(n)}{2^n} = \lim_{n\to\infty} \frac{(\cos n)^2}{2^n} = 0$$

Converge

4) $f(x) = \dfrac{1}{(1+x)^2}$

Note que $\dfrac{1}{(1+x)^2} = (1+x)^{-2}$ com $(1+x)^{-2} = \left(-(1+x)^{-1}\right)' = \dfrac{-1}{1+x}$ → derivada

Então,

$$\frac{-1}{(1+x)} = -\left[1 - x + x^2 - x^3 + x^2 + \dots + x^n(-1)^n + \dots\right]$$
$$= -1 + x - x^2 + x^3 - x^4 + \dots + (-1)^{n+1} x^n + \dots$$

Derivando, obtemos (ambos os lados)

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x - 4x + \dots + (-1)^{n+1} \cdot n\, x^{n-1} + \dots$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n\, x^{n-1}$$

(Série de Maclaurin)

5) $\displaystyle\int_0^1 x \cdot \cos x \, dx$

$f(x) = \cos x \Rightarrow f(0) = 1$
$f'(x) = -\operatorname{sen} x \Rightarrow f'(0) = 0$
$f''(x) = -\cos x \Rightarrow f''(0) = -1$
$f'''(x) = +\operatorname{sen} x \Rightarrow f'''(0) = 0$

Sabemos que $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$

Assim, $x \cdot \cos x = x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} - \dfrac{x^7}{6!} + \dots + (-1)^{n+1} \dfrac{x^{2n+1}}{(2n)!} + \dots$

Integrando

$$\int_0^1 x \cos x = \int_0^1 \left[x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots\right] dx = \frac{x^2}{2} - \frac{x^4}{4 \cdot 2!} + \frac{x^6}{6 \cdot 4!} - \frac{x^8}{8 \cdot 6!} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{8} + \frac{1}{144} - \frac{1}{8 \cdot 6!} + \approx$$

$24 \times 6 = 144$

$6 \cdot 3 \cdot 2 \cdot 1$